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Waves produced by a vertically oscillating plate

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The vertical oscillation of a plate partially immersed in a non-wetting fluid produces a radiated wavetrain when the contact line between the plate and the free surface of the fluid cannot move freely along the plate. Realistic conditions to apply at the contact line when capillarity is not negligible include the dynamic variation of the contact angle and contact-angle hysteresis. Both of these effects are included in this paper and the amplitude of the radiated waves and the energy dissipation at the contact line are calculated.

1. Introduction

The interaction between surface waves and partially immersed bodies, which may be fixed or floating or may be executing forced motions, has been the subject of many investigations. The review article by Wehausen (1971) contains an account of some of this work and the general problem of wave–body interaction is central to many devices for the extraction of energy from waves (see Evans 1981). In many applications the effect of surface tension can safely be ignored and most, if not all, of the theory has treated gravity waves only. When very short waves are important or when gravity is effectively reduced, at the interface between two fluids of nearly equal density, for example, the restoring forces of gravity and capillarity must both be included. There has been some recent work on capillary–gravity waves (Hogan 1979; Vanden-Broeck 1983) in horizontally unbounded regions, but little consideration has been given to the interaction of such waves with vertical boundaries.

A special feature of the inclusion of surface tension is the need to add edge conditions at the intersection of the free surface of the fluid with the boundary. These conditions are required because of the increase in order of the dynamic boundary condition at the interface when surface tension is present. The first discussion of the need to impose edge conditions is by Benjamin & Scott (1979), who argued that the appropriate edge condition is, in many cases, that the contact line remain fixed throughout the motion. Their main interest is in a rim-full channel, for which the fixed contact line is certainly appropriate, but they also argue that the same condition would apply on a solid surface when the contact angle between the fluid and the solid exhibits hysteresis, that is, when there is a range of possible static contact angles. The same condition was applied by Graham-Eagle (1984) in his determination of the frequencies of capillary–gravity waves in a full circular cylinder. Another possible choice of edge condition was proposed by Hocking (1987) in his study of the damping of waves in a region bounded by two vertical walls. The observed contact angles between a fluid and a solid are velocity dependent, and a model of the edge condition that incorporates this behaviour, but ignores hysteresis, is to require that the contact angle be proportional to the relative velocity of the contact line on the surface.

Contact-angle hysteresis is a nearly universal feature of real materials; for surfaces that are very smooth and chemically homogeneous its influence may be relatively small even if it cannot be completely eliminated. The model condition includes as limiting cases both the fixed contact line used by Benjamin & Scott (1979) and Graham-Eagle (1984) and the fixed contact angle, which is the condition obeyed by gravity waves in the presence of vertical walls when there is no surface tension. In the general case, this condition implies the dissipation of energy at the contact line and Hocking (1987) calculated decay rates for standing waves between two vertical walls. This decay is produced partly by processes in the vicinity of the contact line, as modelled by this edge condition, and partly by viscosity; in some cases, the first dissipative process may dominate the second.

Following this treatment of standing capillary–gravity waves, it is natural to consider the wavetrains produced by an oscillating body in a horizontally unbounded region. Without surface tension such waves are produced by the displacement of the fluid by the body. With surface tension present, the restriction on the motion of the contact line on the surface of the body imposed by the edge condition is an additional source of propagating waves on the fluid surface. Since this is a novel feature not present when only pure gravity waves are possible, it is instructive to consider a special case when there is no displacement effect. If a thin vertical plate intersects the free surface of the fluid and is forced to oscillate in its vertical plane no waves will be produced (in an inviscid fluid) if the contact line can move freely up and down the plate. If, however, the contact line is fixed on the plate, or if the contact angle varies with speed, the fluid near the plate will be brought into motion and a wavetrain propagating away from the plate will be produced. The waves produced in this way by an oscillating plate and the corresponding energy balance are calculated in this paper with and without contact-angle hysteresis.

As well as providing information concerning the effect of the combination of surface tension and the edge condition on the interaction between surface waves and moving bodies, the particular model-problem solved has application to the Wilhelmy plate apparatus. This is a means for determining contact angles in dynamic conditions and consists of a thin plate which is slowly lowered into or removed from fluid at rest. From careful measurements of the vertical force on the plate during its motion, variations in the contact angle can be inferred. Young & Davis (1987) have applied a realistic edge condition (including hysteresis) to the oscillatory motion of such a plate. They show that, for parameters applicable to the operation of the apparatus, it is possible to determine the motion of the contact line along the plate, and the predicted force-balance, without having to calculate the waves produced by the motion of the plate. The present results extend the work of Young & Davis to a parameter range when the waves and the contact-angle motion have to be calculated simultaneously.

The major restrictions on the physical situation are that the amplitude of the motion is small enough for the waves to be linearized and that viscosity has a negligible effect. A more severe restriction, and one that is less easy to justify, is that the static contact angle is centred on 90°. This greatly simplifies the analysis, and a similar condition has been used by Graham-Eagle (1984), Hocking (1987) and Young & Davis (1987). A second limitation is that the motion is two-dimensional, so waves propagate only in the direction normal to the plate. The problem to be solved is formulated in §2 and the relevant non-dimensional parameters are identified there. The energy balance is discussed in §3 and an expression from which the rate of

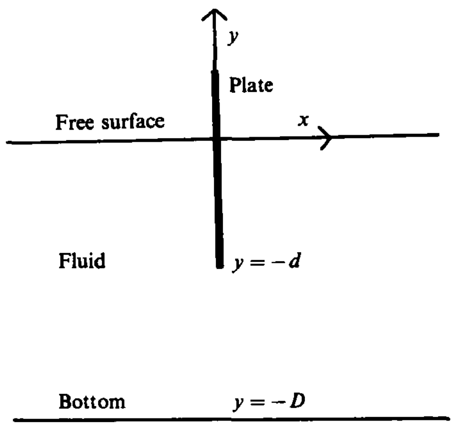

FIGURE 1. The oscillating plate.

dissipation can be calculated is given. The solution without contact-angle hysteresis is in §4. With contact-angle hysteresis present, the analysis is more difficult and a closed-form solution cannot be found; in §5 the problem is formulated as an integral equation which is solved numerically. Some discussion of the results and their comparison with those found by Young & Davis (1987) closes the paper.

2. Formulation and parameters

Consider a thin plate which is oscillating sinusoidally and vertically in its own vertical plane (figure 1). The plate intersects the free surface of fluid of depth D'. The width of the plate is the same as that of the channel containing the fluid but any effect of the lateral boundaries is ignored; the motion is entirely two-dimensional. The bottom edge of the plate has a mean depth d', less than D', and the velocity of the plate has an amplitude V' and an angular frequency σ'. The lengthscale chosen as a basis for non-dimensionalization is proportional to the wavelength $2\pi/k'$ of surface waves having the same frequency as the oscillation of the plate. The coordinates of a point in the fluid are denoted by $(x,y)/k'$, with origin at the intersection of the plate and the undisturbed free surface. The x-axis is horizontal and normal to the plate and the y-axis is vertical and upward.

The corresponding velocity components are $V'(u,v)$, and time is measured by $(gk')^{-\frac{1}{2}}t$, pressure by $\rho V'(g/k')^{\frac{1}{2}}p$ and the free-surface elevation by $V'(gk')^{-\frac{1}{2}}\eta$, where ρ is the uniform density of the fluid and g is the gravitational acceleration. The bottom edge of the plate in its mean position is at $y=-d$ and the bottom of the fluid is at $y=-D$, where $d=k'd'$ and $D=k'D'$. The scaling quantity k' can be determined from the equation for capillary–gravity waves of the given frequency, namely

$$\sigma'^2=\left(gk'+\frac{\gamma k'^3}{\rho}\right)\tanh k'D', \tag{2.1}$$

where γ is the surface tension at the fluid/air interface.

The linearized equations for inviscid fluid motion are, with these scaled variables,

$$\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x},\quad \frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.2}$$

The associated boundary conditions are

$$u = 0 \quad \text{on } x = 0, \quad v = 0 \quad \text{on } y = -D, \tag{2.3}$$

the first condition applying both on the plate and below it, since the motion is symmetrical about the plane $x = 0$. On the free surface, the kinematic and dynamic conditions are that

$$\frac{\partial \eta}{\partial t} = v, \quad \eta - K\frac{\partial^2 \eta}{\partial x^2} = p \quad \text{on } y = 0, \tag{2.4}$$

where

$$K = \frac{\gamma k'^2}{\rho g}. \tag{2.5}$$

The parameter K measures the relative importance of surface tension and gravity and is therefore a Bond number. The velocity of the plate is given by $V'V$, where

$$V = \cos \sigma t, \tag{2.6}$$

and

$$\sigma^2 = \frac{\sigma'^2}{gk'} = (1+K)\tanh D. \tag{2.7}$$

The waves produced by the oscillation of the plate propagate away from the plate so that a radiation condition must be satisfied. The equations permit solutions with asymptotic behaviour as x tends to infinity of the form

$$\eta \sim R_+ \exp\{\mathrm{i}(\sigma_k t + kx)\} + R_- \exp\{\mathrm{i}(\sigma_k t - kx)\}. \tag{2.8}$$

where σ_k is the angular frequency of capillary–gravity waves of length $2\pi/k$ and is given by

$$\sigma_k^2 = k(1+Kk^2)\tanh kD. \tag{2.9}$$

The radiation condition is that

$$R_+ = 0. \tag{2.10}$$

The final condition needed to close the problem is the condition at the contact line. A suitable model for this condition that includes both contact-angle hysteresis and the dynamic behaviour of the contact angle is sketched in figure 2 and has the form

$$\frac{\partial \eta'}{\partial t'} - V'V = \begin{cases} \lambda'_{\mathrm{a}}\left(\dfrac{\partial \eta'}{\partial x'} - \alpha'_{\mathrm{a}}\right) & \text{if } \dfrac{\partial \eta'}{\partial x'} > \alpha'_{\mathrm{a}}, \\ 0 & \text{if } \alpha'_{\mathrm{r}} < \dfrac{\partial \eta'}{\partial x'} < \alpha'_{\mathrm{a}}, \\ \lambda'_{\mathrm{r}}\left(\dfrac{\partial \eta'}{\partial x'} - \alpha'_{\mathrm{r}}\right) & \text{if } \dfrac{\partial \eta'}{\partial x'} < \alpha'_{\mathrm{r}}. \end{cases} \tag{2.11}$$

This condition applies at $x = 0$ on the positive side of the plate; a similar condition with the sign of the right-hand terms reversed holds on the negative side, but the symmetry of the problem implies that we need only consider x positive. When the slope of the free surface lies between α'_{r} and α'_{a} the contact line does not move relative to the plate; when the slope exceeds α'_{a} the contact line advances up the plate and it retreats when the slope falls below α'_{r}. The chosen form of the edge condition is consistent with the known behaviour of contact angles (Dussan V. 1979) and has the same form as that used by Young & Davis (1987); the linear relationship between slope and speed is valid for low speeds. For motions of sufficiently small amplitude,

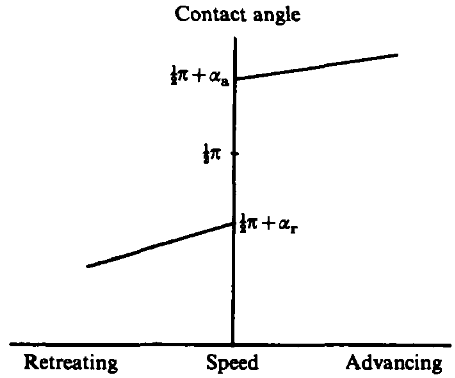

FIGURE 2. The contact-angle model.

the slope will remain within the static range of contact angles and the contact line will remain fixed, as in the problems studied by Benjamin & Scott (1979) and Graham-Eagle (1984). Contact-angle hysteresis will only influence the motion if the amplitude of the oscillation of the plate is large enough for the slope to exceed α'_a or fall below α'_r. The scaled version of (2.11) has the form

$$\frac{\partial \eta}{\partial t} - \cos \sigma t = \begin{cases} \lambda_a \left(\dfrac{\partial \eta}{\partial x} - \alpha_a \right) & \text{if } \dfrac{\partial \eta}{\partial x} > \alpha_a, \\ 0 & \text{if } \alpha_r < \dfrac{\partial \eta}{\partial x} < \alpha_a, \\ \lambda_r \left(\dfrac{\partial \eta}{\partial x} - \alpha_r \right) & \text{if } \dfrac{\partial \eta}{\partial x} < \alpha_r, \end{cases} \tag{2.12}$$

where
$$\lambda'_a = \left(\frac{g}{k'}\right)^{\frac{1}{2}} \lambda_a, \quad \alpha'_a = V' \left(\frac{k'}{g}\right)^{\frac{1}{2}} \alpha_a, \tag{2.13}$$

with similar expressions for λ_r and α_r. In what follows, the general case will be simplified by assuming that

$$\lambda_a = \lambda_r, \quad \alpha_r = -\alpha_a; \tag{2.14}$$

the asymmetrical case can be treated by the method explained in §5, but the method of §4 for the non-hysteresis problem requires the symmetry.

This completes the description of the problem to be solved. The parameters are K, D, λ and α. The depth d of the bottom edge of the plate does not enter the problem in the absence of viscosity since the only boundary condition on the plate, namely $u = 0$, applies also to the fluid below the plate. The primary effect of viscosity is the presence of a Stokes layer on the plate, as in the standing-wave problem of Hocking (1987). The relevant parameter is defined by

$$f = \nu \left(\frac{k'^3}{g}\right)^{\frac{1}{2}}, \tag{2.15}$$

where ν is the kinematic viscosity of the fluid and the viscous dissipation is of order $f^{\frac{1}{2}}$. The neglect of viscosity is therefore valid provided $f \ll 1$. The scalings used by Young & Davis (1987) in their study of the Wilhelmy plate differ from those employed here. They work with two parameters, B and C, which are respectively a Bond number and a capillary number, and for small values of K these parameters can be written in terms of f and K, with

$$B = \frac{f}{K}, \quad C = \frac{f}{K^{\frac{1}{2}}}. \tag{2.16}$$

Young & Davis assume that C is small and that B is arbitrary, which implies that f and K are both small. In the present work, f is small but K can have an arbitrary size.

3. Energy balance

The energy E of the fluid occupying the region $0 < x < X$, $-D < y < 0$ consists of kinetic energy, gravitational potential energy and the energy of the free surface. Integration of the equations of motion and the application of the boundary conditions show that the rate of change of E is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_{-D}^{0} up|_X \,\mathrm{d}y + K\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_X - K\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_0. \tag{3.1}$$

In periodic motion E does not change over a period. The energy E_{R} radiated away from the plate by the waves is given by

$$E_{\mathrm{R}} = \int_0^{2\pi/\sigma}\left[\int_{-D}^{0} up|_X \,\mathrm{d}y - K\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_X\right]\mathrm{d}t, \tag{3.2}$$

and (3.1) shows that this energy is equal to that supplied to the fluid by the plate, that is,

$$E_{\mathrm{R}} = -\int_0^{2\pi/\sigma} K\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\bigg|_0 \mathrm{d}t. \tag{3.3}$$

The external force applied to the plate must account for the mass-acceleration of the plate and its net weight allowing for buoyancy; the remaining part of the force balances the surface forces. Hence the energy E_{S} supplied to the fluid and the surface is given by

$$E_{\mathrm{S}} = -\int_0^{2\pi/\sigma} K\cos\sigma t\frac{\partial\eta}{\partial x}\bigg|_0 \mathrm{d}t. \tag{3.4}$$

The equation for the energy balance is then

$$E_{\mathrm{S}} = E_{\mathrm{R}} + E_{\mathrm{D}}, \tag{3.5}$$

where E_{D} is the energy dissipated at the edge and is given by

$$E_{\mathrm{D}} = \int_0^{2\pi/\sigma} K\left(\frac{\partial\eta}{\partial t} - \cos\sigma t\right)\frac{\partial\eta}{\partial x}\bigg|_0 \mathrm{d}t. \tag{3.6}$$

This energy dissipation is inherent in any model that allows for dynamic contact-angle behaviour (see Dussan V. & Davis 1986) and arises from the structure of the motion in the vicinity of the contact line. The energies defined here are all for one period of the motion and for unit width in the spanwise direction. There is no dissipation of energy if the contact angle is fixed, which can happen in the limit as $\lambda \to \infty$, nor if the contact line is fixed, which corresponds to $\lambda = 0$ or to large hysteresis.

The radiated energy E_{R} can be calculated in terms of the amplitude of the radiated waves. If, for large x,

$$\eta \sim c\exp\{\mathrm{i}(\sigma t - x)\}, \tag{3.7}$$

the corresponding value for the pressure can be written as

$$p \sim c(1+K)\frac{\cosh(y+D)}{\cosh D}\exp\{\mathrm{i}(\sigma t - x)\}, \tag{3.8}$$

and from (3.2) we can evaluate this energy in the form

$$E_{\mathrm{R}} = \frac{\pi Q}{2\tanh D}|c|^2, \tag{3.9}$$

where

$$Q = (1+3K)\tanh D + (1+K)\,D\,\mathrm{sech}^2 D. \tag{3.10}$$

4. No contact-angle hysteresis

Although real materials exhibit contact-angle hysteresis to some extent, it is helpful to consider first the motion when there is no hysteresis. This covers the case when the stick phase of the motion occupies a negligible fraction of the oscillatory motion. It also includes (by setting $\lambda = 0$) the opposite extreme, when the hysteresis is so large that the edge does not move relative to the plate, and, with $\lambda = \infty$, the free-end case appropriate for pure gravity waves, which is useful for comparison. In this special case of no contact-angle hysteresis, the edge condition (2.12) has the simpler form

$$\frac{\partial \eta}{\partial t} - \cos\sigma t = \lambda\frac{\partial \eta}{\partial x}. \tag{4.1}$$

The motion is now sinusoidal and can be found by an extension of the method used by Hocking (1987) in the standing-wave problem. In order to accommodate phase changes we can replace the term $\cos\sigma t$ in the edge condition (4.1) by $\exp(\mathrm{i}\sigma t)$, and then this time factor can be removed from all the dependent variables. The pressure and the velocity components can be written as Fourier integrals in the form

$$p = \int_0^\infty P(k)\cos kx\frac{\cosh k(y+D)}{\cosh kD}\,\mathrm{d}k, \tag{4.2}$$

$$\mathrm{i}\sigma u = \int_0^\infty kP\sin kx\frac{\cosh k(y+D)}{\cosh kD}\,\mathrm{d}k, \tag{4.3}$$

$$\mathrm{i}\sigma v = -\int_0^\infty kP\cos kx\frac{\sinh k(y+D)}{\cosh kD}\,\mathrm{d}k, \tag{4.4}$$

which satisfy the equations of motion and the conditions at $x = 0$ and at $y = -D$. The surface elevation can be found from the dynamic condition at the free surface and has the form

$$\eta = \int_0^\infty \frac{P}{1+Kk^2}\cos kx\,\mathrm{d}k + \tfrac{1}{2}\pi K^{-\frac{1}{2}}A\exp(-xK^{-\frac{1}{2}})$$

$$= \int_0^\infty \frac{P+A}{1+Kk^2}\cos kx\,\mathrm{d}k, \tag{4.5}$$

where A is an arbitrary constant. The kinematic condition at the free surface gives another expression for η, namely

$$\mathrm{i}\sigma\eta = -\frac{1}{\mathrm{i}\sigma}\int_0^\infty kP\tanh kD\cos kx\,\mathrm{d}k, \tag{4.6}$$

and, on equating these two values for η, we obtain

$$P = \frac{A}{k(1+Kk^2)\tanh kD - \sigma^2} + (1+K)\,c\delta(k-1), \tag{4.7}$$

where c is an arbitrary constant and δ is the Dirac delta function.

The asymptotic value of η as $x \to \infty$ follows from (4.6) and (4.7). The pole at $k = 1$ gives the dominant contribution to the principal value of the integral and we have

$$\eta \sim c\cos x - \frac{\pi \tanh D}{Q} A \sin x. \tag{4.8}$$

With the time factor re-introduced, the radiation condition gives

$$\eta \sim c\exp\{\mathrm{i}(\sigma t - x)\}, \tag{4.9}$$

provided that

$$A = \frac{\mathrm{i}Q}{\pi \tanh D} c. \tag{4.10}$$

The edge condition (4.1) and this relation between the values of A and c then show that

$$c = \frac{-1}{C_\mathrm{r} - \mathrm{i}C_\mathrm{i}}, \tag{4.11}$$

where

$$C_\mathrm{r} = \frac{Q}{\pi\sigma} ⨍_0^\infty \frac{k\tanh kD}{k(1+Kk^2)\tanh kD - \sigma^2}\,\mathrm{d}k, \tag{4.12}$$

$$C_\mathrm{i} = \frac{1+K}{\sigma} + \frac{\lambda Q}{2K}. \tag{4.13}$$

The radiated and supplied energies can be calculated from (3.3) and (3.4) and after some simplification we obtain their values in the form

$$E_\mathrm{R} = \frac{\pi Q}{2\tanh D}\frac{1}{C_\mathrm{r}^2 + C_\mathrm{i}^2}, \tag{4.14}$$

$$E_\mathrm{S} = \frac{\pi Q}{2\sigma}\frac{C_\mathrm{i}}{C_\mathrm{r}^2 + C_\mathrm{i}^2}. \tag{4.15}$$

The integral in the equation for C_r can be evaluated by contour integration. Standard methods give the value of the integral in the form

$$⨍_0^\infty \frac{k\tanh kD}{k(1+Kk^2)\tanh kD - \sigma^2}\,\mathrm{d}k$$

$$= \pi \sum_{n=1}^{\infty} \frac{\mu_n \tan \mu_n D}{(3K\mu_n^2 - 1)\tan\mu_n D + \mu_n(K\mu_n^2 - 1)D\sec^2\mu_n D}, \tag{4.16}$$

where μ_n is the real root of the equation

$$\mu(K\mu^2 - 1)\tan\mu D = (1+K)\tanh D. \tag{4.17}$$

Numerically determined values for E_S, E_R, $E_\mathrm{D} = E_\mathrm{S} - E_\mathrm{R}$ and the amplitude of the radiated wave $|c|$ for various values of K, D and λ are shown in figures 3 and 4. The curves for $D = 1$ are only a little different from those for $D = 10$, indicating that the surface motion only depends significantly on the depth of the fluid when $D < 1$.

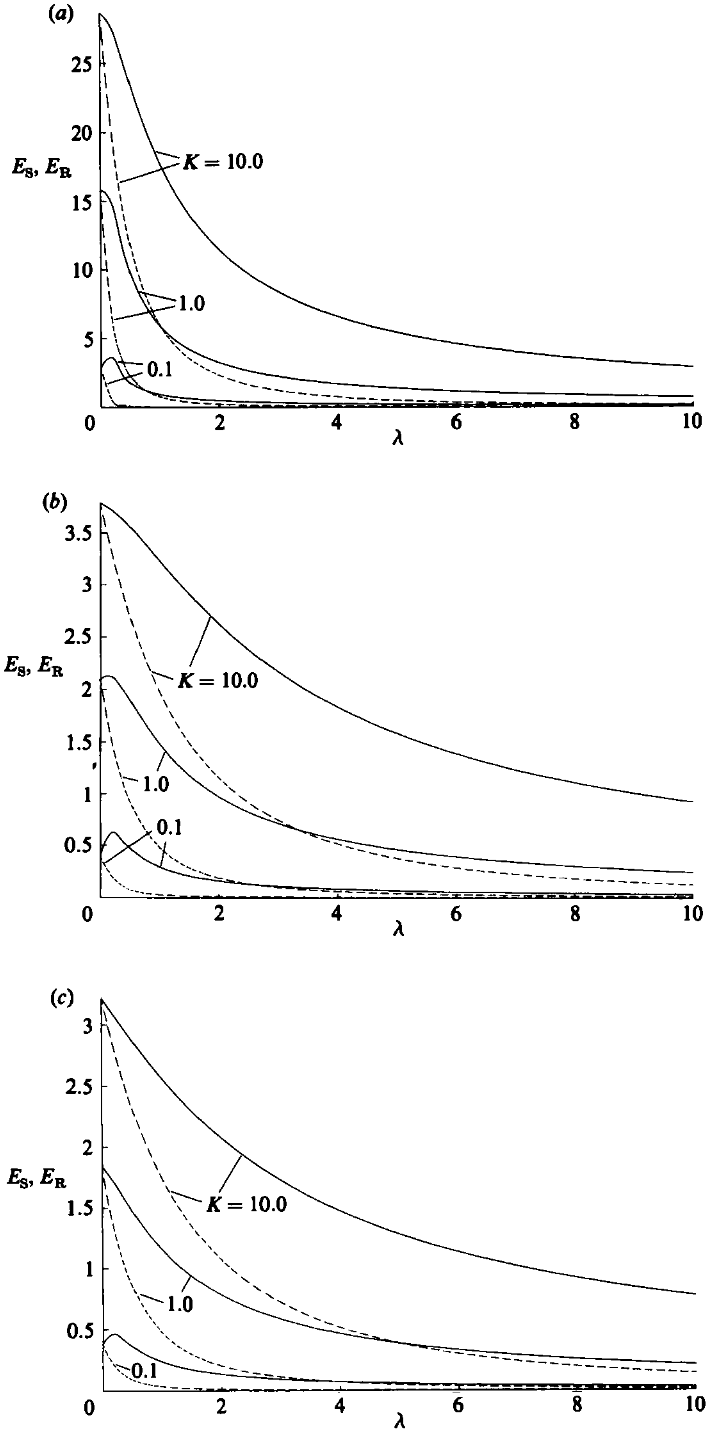

FIGURE 3. Values of the supplied energy E_S (full line) and radiated energy E_R (broken line) for $K = 0.1, 1.0, 10.0$. (*a*) $D = 0.1$, (*b*) $D = 1.0$, (*c*) $D = 10.0$.

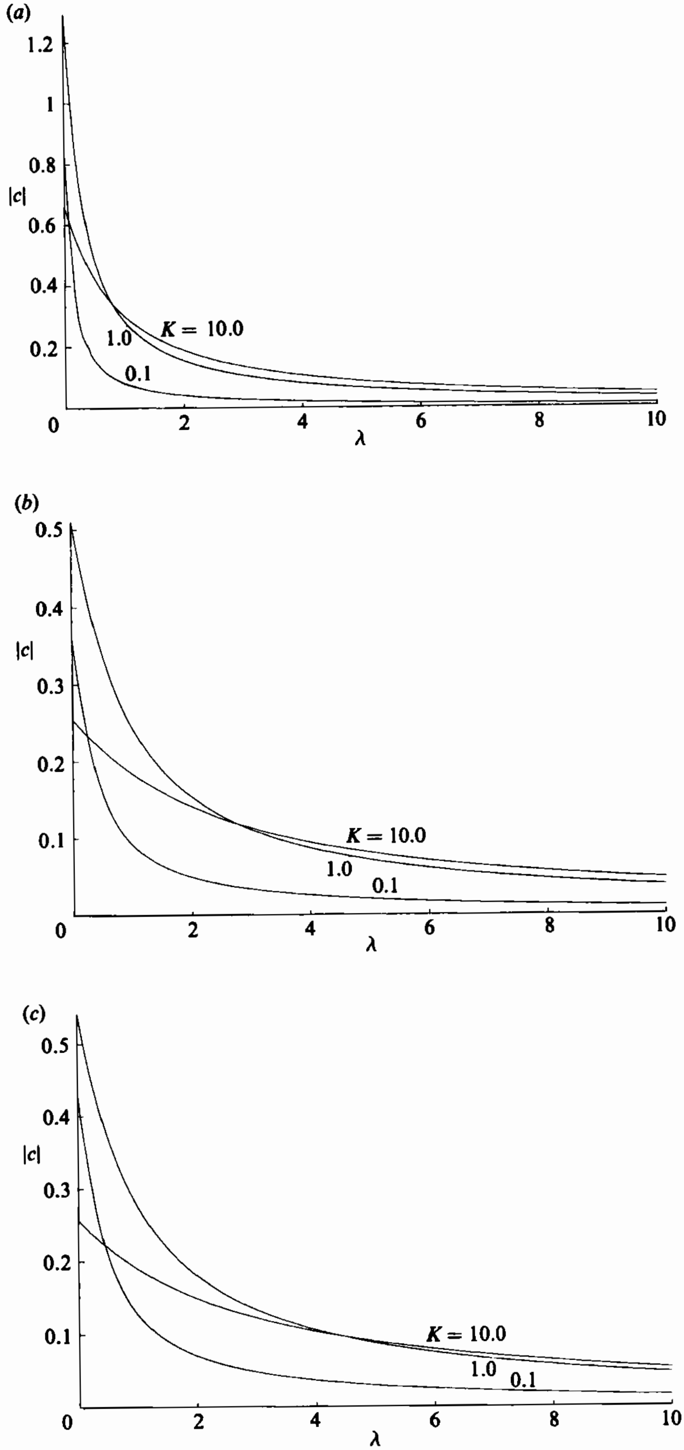

FIGURE 4. Values of the amplitude of the radiated wave for $K = 0.1$, 1.0, 10.0. (*a*) $D = 0.1$, (*b*) $D = 1.0$, (*c*) $D = 10.0$.

For infinite depth the integral in (4.12) can be evaluated exactly, and the corresponding results are that

$$C_r = \frac{(1+K)^{\frac{1}{2}}}{2\pi}\left\{\ln\left(\frac{K+1}{K}\right)+\frac{2(3K+2)}{K^{\frac{1}{2}}(3K+4)^{\frac{1}{2}}}\tan^{-1}\left(\frac{3K+4}{K}\right)^{\frac{1}{2}}\right\}, \quad (4.18)$$

$$C_i = (1+K)^{\frac{1}{2}}+\frac{\lambda(1+3K)}{2K}, \quad (4.19)$$

$$E_R = \frac{\pi(1+3K)}{2(C_r^2+C_i^2)}, \quad E_S = \frac{C_i}{(1+K)^{\frac{1}{2}}}E_R. \quad (4.20)$$

As $\lambda \to \infty$, both E_S and E_R tend to zero because the contact line can slip freely along the surface of the plate, and the fluid remains at rest. For $\lambda = 0$, $E_S = E_R$ and $E_D = 0$, since the contact line is fixed on the plate and there is therefore no dissipation of energy at the contact line. The contact line also remains fixed when the contact-angle hysteresis is sufficiently large for the slope of the free surface to remain within the static range. Hence the results obtained here for $\alpha = 0$ and $\lambda = 0$ also apply when α is large. The solution for arbitrary values of the hysteresis angle α is the next topic to be discussed.

5. Stick–slip motion

Contact-angle hysteresis produces a stick–slip motion of the contact line along the plate. Intervals during which the contact line moves with the plate are interspersed with periods of relative motion in either direction. When the amplitude of the motion of the plate is kept fixed and when λ is not zero, an increase in α produces a decrease in the proportion of the time during which slip occurs, until, for sufficiently large α, there is no slip and the contact line remains fixed to the plate. The two phases of motion prevent the application of the previous method, because the motion, though periodic, is not sinusoidal. The approach used here is to find the evolution of the motion from an initially static state and to continue until the transients have disappeared and a periodic solution has developed.

The equations to be solved are the same as in §2, with the edge condition now of the form

$$\frac{\partial \eta}{\partial t}-\sin\sigma t = \begin{cases} \lambda\left(\dfrac{\partial \eta}{\partial x}-\alpha\right) & \text{if } \dfrac{\partial \eta}{\partial x} > \alpha, \\ 0 & \text{if } \left|\dfrac{\partial \eta}{\partial x}\right| < \alpha, \\ \lambda\left(\dfrac{\partial \eta}{\partial x}+\alpha\right) & \text{if } \dfrac{\partial \eta}{\partial x} < -\alpha. \end{cases} \quad (5.1)$$

The change in the phase of the forcing velocity is purely for convenience and it allows the initial state of the fluid and free surface to be given by

$$p = u = v = \eta = 0. \quad (5.2)$$

A solution of the same form as that used in §3 is possible, and we can write

$$p = \int_0^\infty P(k,t)\cos kx \frac{\cosh k(y+D)}{\cosh kD}\,\mathrm{d}k. \quad (5.3)$$

If we take a Laplace transform in t, and denote the transform of any quantity f by $\bar{f}$, the kinematic condition at the free surface gives the value of the transformed surface elevation in the form

$$\bar{\eta} = -\int_0^\infty \frac{kP}{s^2} \tanh kD \cos kx \, \mathrm{d}k. \tag{5.4}$$

The dynamic free-surface condition shows that

$$\bar{\eta} = \int_0^\infty \frac{\bar{P}}{1+Kk^2} \cos kx \, \mathrm{d}k - K^{\frac{1}{2}}\bar{B} \exp(-xK^{-\frac{1}{2}}), \tag{5.5}$$

where $B(t)$ is the slope of the free surface at the plate. Equating the two expressions for $\bar{\eta}$ and inverting the transform, we can obtain a convolution integral for $\partial\eta/\partial t$ in the form

$$\frac{\partial \eta}{\partial t} = -\int_0^t F_1(\tau, x)\, B'(t-\tau)\, \mathrm{d}\tau, \tag{5.6}$$

where
$$F_1(\tau, x) = \frac{2K}{\pi}\int_0^\infty \frac{k \tanh kD}{\omega} \sin \omega t \cos kx \, \mathrm{d}k, \tag{5.7}$$

and
$$\omega^2 = k(1+Kk^2)\tanh kD. \tag{5.8}$$

The edge condition (5.1) then gives an evolutionary integral equation for B in the form

$$-\int_0^t F(\tau)\, B'(t-\tau)\, \mathrm{d}\tau - \sin \sigma t = \begin{cases} \lambda(B-\alpha) & \text{if } B > \alpha, \\ 0 & \text{if } |B| < \alpha, \\ \lambda(B+\alpha) & \text{if } B < -\alpha, \end{cases} \tag{5.9}$$

where
$$F(\tau) = F_1(\tau, 0), \tag{5.10}$$

and
$$B(0) = 0. \tag{5.11}$$

The asymptotic values of F are given by

$$F(\tau) \sim \frac{2}{\pi}\Gamma(\tfrac{4}{3})\, K^{\frac{1}{3}}\tau^{-\frac{1}{3}} \quad \text{for } \tau \ll 1, \tag{5.12}$$

$$F(\tau) \sim -\frac{8}{\pi}K\tau^{-3} \quad \text{for } \tau \gg 1, \tag{5.13}$$

and the solution of (5.6) for small values of t has the form

$$B(t) \sim -1.091\ldots K^{-\frac{1}{3}}(1+K)^{\frac{1}{2}}t^{\frac{4}{3}}. \tag{5.14}$$

The general solution can be found by expressing the integral over a small time interval δt in terms of the values of B and F at the ends of the interval. Hence at time $N\delta t$ a linear equation can be obtained from which the value of $B(N\delta t)$ can be expressed in terms of the values of $B(n\delta t)$ for $n = 1, 2, \ldots, N-1$. The appropriate right-hand side at this time is determined by solving for all three forms and choosing the one that gives a consistent value of $B(N\delta t)$. Since the values of both F and B for small t are singular, a modified finite-difference approximation was used for the first and last interval in order to take account of the singularities.

The values of B for successively increasing values of t were found by this procedure. The size of δt was varied to check that sufficient accuracy was being

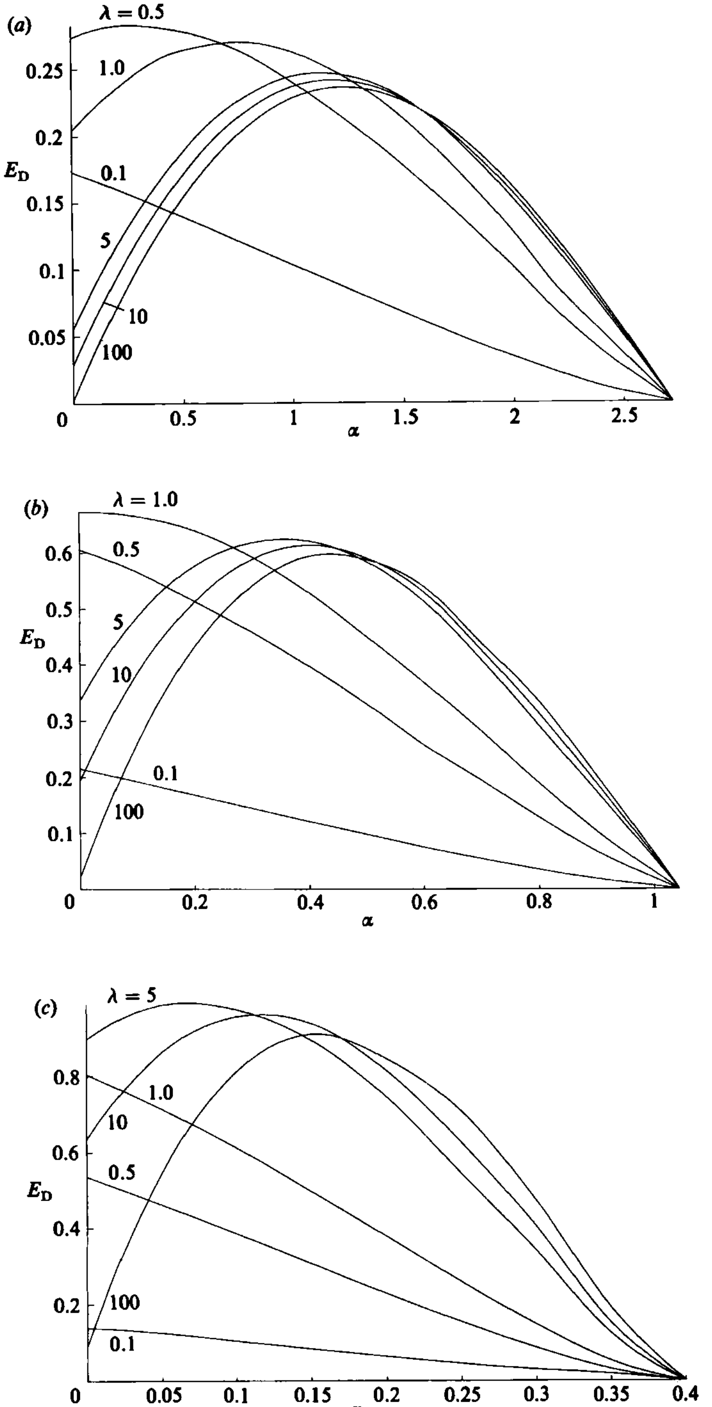

FIGURE 5. Values of the dissipation energy E_D with contact-angle hysteresis and fluid of infinite depth. (*a*) $K = 0.1$, (*b*) $K = 1.0$, (*c*) $K = 10.0$.

obtained (at least three significant figures). The solution proceeded until $B(t)$ was found to be periodic within a small tolerance; in some cases more than 1000 timesteps were needed, but usually 100–200 were sufficient.

Some results for infinite depth are shown in figure 5. The energy dissipation in one period for various values of K and λ is shown for increasing values of α. Beyond a critical value there is no dissipation because the contact line has become fixed. The critical values of α are approximately proportional to $K^{-\frac{1}{2}}$, the ratio varying from about 1 for small K to about 1.5 for large K. It is clear from the figure that the critical values of α are independent of λ.

Because the relevant parameters occur multiplicatively in the edge condition, it is difficult to separate the effect of the dynamic variation from that of the hysteresis. Some indication of their relative importance can be deduced from figure 5. For fixed λ, the change in the energy dissipation can be seen as the hysteresis increases from zero until it is so large that the contact line remains fixed. For fixed α, we can trace the effect of reducing the dynamic variation from its greatest value at $\lambda = 0$ to zero as $\lambda \rightarrow \infty$.

The energy dissipation is a decreasing function of α when λ is small. For larger values of λ the dissipation first increases with α before decreasing to zero at the critical value of α. This is a surprising result, since the increase in the amount of hysteresis lengthens the proportion of the time during which the contact line is fixed and no energy is dissipated. The initial increase in dissipation may be due to a phase shift between the velocities of the contact line and of the plate. This increase in dissipation was also noted by Young & Davis (1987).

6. Conclusions

The methods used in this paper could easily be generalized to include arbitrary plate motions. It is not clear that the removal of the restriction to contact angles that are close to 90° can lead to a tractable problem. The solutions obtained here may give at least a qualitative understanding for general angles, but are likely to be of little use for contact angles that are close to 0° or 180°.

There is a two-fold purpose to the investigations reported in this paper. The first is to show that the generation of waves by an oscillating body depends to a significant extent on the condition applied at the intersection of the free surface and the body when surface tension is not negligible. This is made evident by the choice of a vertical plate, since without surface tension and with the usual freely-moving contact line no waves are generated. The amplitude of the radiated waves has been determined for both fixed and moving contact lines.

The second purpose is to present a solution of a fluid dynamics problem that includes the effect of contact-angle hysteresis. This phenomenon has been known for some time but quantitative results incorporating it are rare. Dussan V. & Chow (1983) and Dussan V. (1985) have discussed the effect of a maximum static contact angle on the shape of a drop on an inclined plane before it begins to slide. The first paper known to me that includes quantitative results for contact lines that reverse their direction of motion is the study of the Wilhelmy plate by Young & Davis (1987). However, in their case the motion of the contact line could be found without reference to the fluid motion. The solution given here requires the contact-line motion and the fluid flow to be calculated simultaneously.

The problem treated by Young & Davis, as already noted, requires the parameter

K used in this paper to be small. If the value of x is rescaled by a factor $K^{-\frac{1}{2}}$, the pressure p is $O(K^{\frac{1}{2}})$ so that the dynamic free-surface condition (2.4*b*) can be solved for η to leading order with $p = 0$. The solution to this order can then be found from the edge condition only, the equation corresponding to (5.9) being

$$-\frac{\mathrm{d}B}{\mathrm{d}t}-\sin\sigma t = \begin{cases} \hat{\lambda}(B-\hat{\alpha}) & \text{if } B > \hat{\alpha}, \\ 0 & \text{if } |B| < \hat{\alpha}, \\ \hat{\lambda}(B+\hat{\alpha}) & \text{if } B < -\hat{\alpha}. \end{cases} \tag{6.1}$$

The change in the scaling of x also changes the values of the parameters λ and α.

There are a number of other problems involving the interaction of waves and solid boundaries which are affected by velocity-dependent contact angles and contact-angle hysteresis. These include the reflection and transmission of an incident wave by a partially submerged body and the waves generated by an oscillating cylinder. A variant of the Wilhelmy plate apparatus uses a small circular cylinder which is lowered into and raised from a pool of fluid. In this case the radiated waves are axisymmetric and not plane, which is a major difference from the waves produced by a vertical plate. Also, there are now two causes for their generation, the contact-line behaviour that has been studied here and the displacement effect that also occurs when the contact line can move freely on the surface of the cylinder.

Another problem of interest is the effect of hysteresis at the sidewalls of a channel along which a wave is propagating. The change in the frequency when the edge is fixed was found theoretically and experimentally by Benjamin & Scott (1979). Stick–slip motion of the edge may produce cross-waves as well as dissipating energy.

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